Algebra Round Solutions Manual

1. Since we are trying to find the number of Size B bottles sold, let b represent this. The total profit is the cost times the number of bottles sold for each bottle size, represented by this equation:
$$3.25(3) + 2.19b = 9.75$$
Solving for b, we find that **0** bottles of Size B were sold.

2. Continuing the sequence defined in the problem, we get six numbers: $1, 4, 16, 64, 256, 1024$. Adding these together we get **1365**.

3. We know that $f(x) = x^2 - 16$ has a vertex at the y-axis ($x = 0$). We also know that the vertex is the minimum for any concave-up parabola. Setting $x = 0$, we find that $f(x)$ has a minimum value of **-16**.

4. When the ball lands it will have a height of zero, so $12t - 5t^2 + 32 = 0$. Rearranging everything to the other side gives us $5t^2 - 12t - 32 = 0$. This is a quadratic that can be factored to $(5t + 8)(t - 4) = 0$. That means $t = 4$ or $t = \frac{-8}{5}$. Time cannot be negative, so the answer is **4**.

5. Use a systems of equations:
$$5x + 7y + 9z = 120$$
$$2x + 6y + 8z = 96$$
$$2x + 3y + 4z = 52$$
Multiply the last equation by 2, yielding $4x + 6y + 8z = 104$. Subtract this from the first equation to get $x + y + z$ = **16**

6. Max's time is 103 seconds/lap. Derek's time is 86 seconds/lap. To find the number of laps that Max runs before being passed:86/(103-86) = 5.06 or **5 full laps**

7. Let a, b, c be the integers with $a > b > c$. Given this, we can write $a + b = 77$ and $bc = 494$. The prime factorization of $494 = 2 \cdot 13 \cdot 19$. Since c is the smallest integer, let $c = 13$ and $b = 2(19) = 38$. c cannot equal 2 because $13(19)$ is bigger than 57. Plugging in b: $a + 38 = 77$, which means $a = 39$. Our three integers end up being 39, 38, 13, and $39 - 13$ =**26**

8. Factoring the numerator, we get $\frac{abcd(a+b+c+d)}{ab+ac+ad+bc+bd+cd}$. Using vieta's formulas, we know that abcd = e/a, a+b+c+d = -b/a, and the denominator equals c/a. Plugging in the numbers from the given equation, we get -5406/48, or **-901/8**

General Round Solutions Manual

1. Solution 1: Brute force: $1 + 2 + 3 + 4 + 8 + 6 + 12 + 24 = 60$

   Solution 2: Well-known Formula: Given an integer $n = p_1^{x_1} p_2^{x_2}$...where p_i is a prime factor of n, then $(p_1^0 + p_1^1 + p_1^2 + ... p_1^{x_1})(p_2^0 + p_2^1 + p_2^2 + ... p_2^{x_2})$...is equivalent to the sum of the distinct divisors of n. Using this formula, we get $(1 + 2 + 2^2 + 2^3)(1 + 3) = 60$.

2. Notice that we can't guarantee two identical socks when you pull out 100 socks because each could be distinct. Thus, if we pull one more we can ensure this, giving **101 socks.**

3. Use an arithmetic sequence (7,14,21,28...196) defined by $7x$, we see that there are 28 multiples of 7 less than 200. Now, we find the sum of this arithmetic sequence by using 28

4. Count the number of serial numbers that have a digits sum of 6 or less, then subtract this number from the total number of possible serial numbers (10*9*8*7=5040). Note that the only way to achieve a distinct sum of 4 or less is (0,1,2,3), this can be permuted 24 ways, so the answer is 5040-24=5016

5. One solution is to simply guess and check by checking one more than factors of 17, which would be a fairly quick solution and would get 52. However, the more efficient way (if we had more complex/larger numbers) is to use mod systems of 3 mod 7, 2 mod 5, and 1 mod 17

$x \equiv 3 \, mod \, 7 \; (x = 7a + 3)$

$x \equiv 2 \, mod \, 5 \; (x = 5b + 2)$

$x \equiv 1 \, mod \, 17 \; (x = 17c + 1)$

It follows that $7a + 1 = 5b$, so $2a$ is congruent to -1 mod 5, which leads to $2x \equiv 34 \, mod \, 35$, applying this method again with the last modular congruence, we get $x = 52$.

6. There are $5 \, choose \, 2 = 10$ways to choose the remaining two people of the council after we guarantee that both freshmen are in the council. There are $7 \, choose \, 4 = 35$total ways to choose a 4-person council with no restrictions. Our answer is thus $\frac{10}{35} = \frac{2}{7}$.

7. Notice that the units digit of $2017^x$is $1, 7, 9, 3$when$x \equiv 0, 1, 2, 3 \; mod \; 4$, respectively. Now, we must evaluate $2017^{2017} mod \; 4$. By *Fermat's Little Theorem*, since $2017$and$4$are relatively prime, $2017^{2016} \equiv 1 \; mod \; 4, \; so \; 2017^{2017} \equiv 2017 \equiv 1 \; mod \; 4$. Therefore, the units digit is $7 - 1 = 6$.

8. There are $10^3 = 10,000$different combinations for $a_1a_2a_3$. Therefore, $a_4a_5a_6$and $a_5a_6a_7$each have $10,000$combinations. However, notice that $a_1a_2a_3$can be identical to both$a_4a_5a_6$and $a_5a_6a_7$, so we have to subtract 10combinations since they are counted twice. We get 10because the only way for all three to be identical is if $a_1, a_2,... \; a_7$are all the same digit. The answer is thus $2(10,000) - 10 = 19990$.

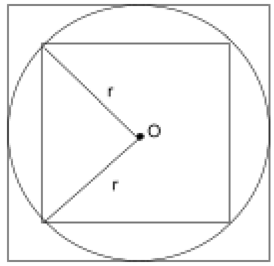
Solutions for Geometry Problems

1. 14/2 = 7. This means that the length+width of the fence is equal to 13. Since we know the length, the width is equivalent to 7-4=3 feet which is 3*12 = 36 inches. The diagonal length is thus $\sqrt{48^2 + 36^2} = 60 \; inches$

2. Pythagorean Theorem implies that the other side must be 5, which means the triangle is isosceles and right, so the answer is 45 degrees.

3. The diagonal of a square given a side s can be represented as $s\sqrt{2}$. It follows that the diagonal of the base is equivalent to $4\sqrt{2} * \sqrt{2} = 8$ inches. It follows by Pythagorean Theorem that the spatial diagonal is $\sqrt{8^2 + 6^2} = 10$.

4. Note that between two tangent circles, the centers and the point of tangency are collinear. Therefore, the sides of the triangle are 7, 8, and 9. You can now apply Heron's formula which states that given the semiperimeter, s, of a triangle, its area is equivalent to $\sqrt{s(s - a)(s - b)(s - c)}$where a, b, and c are the side lengths of the triangle and $s = \frac{a+b+c}{2}$. The answer is $\sqrt{12(5)(4)(3)} = 12\sqrt{5}$



5. For the inscribed circle, we can form an isosceles triangle. That means angle O is right, and due to the pythagorean theorem the side length of the inscribed circle is $r\sqrt{2}$. For the circumscribed square, we draw a diameter that is colinear and parallel to two sides of the circumscribed square. Since the diameter has endpoints that are perpendicular to the sides of the square its length is equal to the side of the square. That means the large square has a side length of $2r$. Therefore, the ratio of the areas is $\frac{(2r)^2}{(r\sqrt{2})^2} = \frac{4r^2}{2r^2}$ =2.

6. BC must be a diameter as are AC and AB. The two semicircles have combined area $\frac{1}{2}(4\pi + \frac{9}{4}\pi)$The area between semicircle BAC and triangle ABC is $\frac{1}{2}(\frac{25}{4}\pi) - 6$. So shaded area is $\frac{1}{2}(4\pi + \frac{9}{4}\pi) - (\frac{1}{2}(\frac{25}{4}\pi) - 6) = 6$

7. Notice that $AB$is the radical axis of the two circles, which means $O_1O_2$is the perpendicular bisector of AB. Let their intersection be X. Thus, by Pythagorean Theorem, $OX = 4$, so $XD = 9$. We apply Pythagorean Theorem again to obtain $AD = \sqrt{90} = 3\sqrt{10}$.

8. Reflecting B about DC to B’ gives isosceles trapezoid (and cyclic quad) ACB’D. Where Angle ADB’ = 88 degrees. It can be also be found that angle BDC = 180-(88+72)=20 degrees and that angle ADB = 88-2*20=48 degrees. Thus, angle ACD = 180-(92+48+20)=20 degrees.

Team Round Solutions Manual

1. Notice that we must have each squared value equal to 0 (or else we will never be able to obtain a sum of zero due to a lack of negative terms). Therefore, $a = 3,\ b =- 4,\ c = 12,\ d =- 5,\ and\ (3)(-\ 4) - (12)(-\ 5) = 48$.

2. There are only three different types of triangles that we can consider as candidates for having the largest area: edge-edge-face diagonal, edge-diagonal-spatial diagonal, and an equilateral triangle with face diagonal edges. We get areas of $50,\ 50\sqrt{2},\ 50\sqrt{3}$, respectively. Therefore, the largest area is $50\sqrt{3}$.

3. By Vieta's Formula, $abcd\ =\ \frac{9}{4},\ a\ +\ b\ +\ c\ +\ d = \frac{-11}{4},\ \frac{abcd}{a+b+c+d} = \frac{-9}{11}$.

4. We can split this problem into 4 cases:
   4 Good Songs: $(7\ choose\ 4)(8\ choose\ 3)\ =\ 1960$
   5 Good Songs: $(7\ choose\ 5)(8\ choose\ 2)\ =\ 588$
   6 Good Songs: $(7\ choose\ 6)(8\ choose\ 1)\ =\ 56$
   7 Good Songs: 1way
   The sum is **2605**

5. The units digit of exponents with base 7 follow the repeating pattern 7,9,3,1. Thus, 7^x must have a units digit of 3, and 7^{x+3} must have a units digit of 9. The units digit of the sum must be **2**

6. There are 8 choose 4 ways to choose the number of ways that we can get 4 heads from 8 flips. Each head has a probability of ⅓ and each tail has a probability of ⅔, so the answer is$(8\ choose\ 4)(\frac{1}{3})^4(\frac{2}{3})^4 = (70)(1/81)(16/81) = 1120/6561.$

7. Note that $OX = 5$ (by Pythagorean Theorem) and perpendicularly bisects EF. Let XO intersect $EF$and $A.EA$is the height of triangle $XED$and has length$\frac{2*(3*4/2)}{5} = \frac{12}{5}$. Thus, $EF = \frac{24}{5}$.

8. Notice that $\frac{x^2-x}{7} = x,\ so\ x^2 - 8x = x(x-8) = 0$. The solutions to this equation are 0 and 8,
   but 0 is extraneous. **8** is the correct answer

9. From the probability given, the number of purple balloons after adding 12 must be a multiple of 19. Immediately, we can tell that there must be 19 purple balloons after the addition. Thus, the total number of balloons is **62**

10. (Omitted)By cyclic properties, angle ADC=60 degrees. By well known geometric properties, we know that angle ACB' = ADC = **60**.

Finalists' Round Solutions Manual

1. Let $x = a+1,\ y = b+1,\ z = c+1$. This will guarantee that x, y, z are positive integers in our next step. We now have $a + b + c = 8$, where a, b, c are non-negative integers. Applying the stars and bars method, we get $10\ Choose\ 2 = 45$. For more on this "stars and bars" combinatorial method, see the AoPS article on it. As a general rule of thumb, when you are distributing n items to x individuals who must receive at least 0 items, there are $(n + x - 1)\ Choose\ (x - 1)$ways to do so.

2. By the Pythagorean Theorem, we have $(x+2)^2 + 12^2 = (x+4)^2$, which simplifies to $4x = 132$, so $x = 33$.

3. This is equivalent to finding the largest difference between two distinct x-coordinates or y-coordinates. The largest x-coordinate difference is $26 - (-15) = 41$while the largest y-coordinate difference is $18 - (-54) = 72$. Thus the side length of the smallest square to contain these points is 72.

4. Answer: 0

5. Justification for #4: Notice that $9^x \equiv (-1)^x \, mod\, 10$. When x is even, $9^x \equiv 1 \, mod\, 10$ and when x is odd, $9^x \equiv \; -1 \, mod\, 10$. Since parity is different between consecutive integers, $9^x + 9^{x+1} \equiv 0 \, mod\, 10$for all non-negative integers x. (Similar solutions will be given full credit even if mods are not mentioned) Or, since $9^x + 9^{x+1}$factors to $9^x(1 + 9^1)$this is equivalent to 9^x(10)

6. We can express Max's movement as a series of 5 left moves and 7 upwards moves. Our problem is now to find the number of distinct arrangements of these moves. For example, LLLLLUUUUUUU and LUULUULULLUU are two possible paths. There are $12 \, Choose\, 5 = 792$ distinct paths.

7. The height of the trapezoid must be $\frac{48}{(4+8)/2} = 8$. Therefore, $DG = 4$. Since the projection of Eonto $AD$is located halfway between $A$and D, $EG = 2$. Therefore, by Pythagorean Theorem, $DE = \sqrt{20} = 2\sqrt{5}$.(Both forms are accepted)

8. Let $a = x + \frac{1}{x}$. Notice that $x^2 + 3x + 4 + \frac{3}{x} + \frac{1}{x^2} = 0 = a^2 + 3a + 2 = (a + 1)(a + 2)$. From here, we have that $x^2 + x + 1 = 0 \, and \, x^2 + 2x + 1 = 0$where the first polynomial has no real solutions and the second is equivalent to $(x + 1)^2$, which has one real solution, namely -1. So the answer is 1.

9. This probability is equivalent to $(100 \, choose\, 3) \; * \; (12/34)^{97} * (22/34)^3$. When simplified, this is equivalent to $(161700) * (6/17)^{97} * (11/17)^3$, so the answer is $161700 + 1 + 6 + 17 + 97 + 11 + 17 + 3 = 161852$.

10. Extend $BC$and$AD$to meet at point $E$and notice that $ECD$is a $30 - 60 - 90$right triangle similar to EAB. From their properties, we obtain that $EA = 20$. Therefore, by similar triangles, $CD = \frac{20+26}{\sqrt{3}} = \frac{46\sqrt{3}}{3}$.By Pythagorean Theorem, $AC = \sqrt{26^2 + \frac{46^2}{3}} = \frac{\sqrt{4144}}{3}$